Delaunay Mesh Reconstruction From 3D Medical Images Based On Centroidal Voronoi Tessellations

Sizhe Lv*, Xubo Yang*, Lixu Gu†, Xiaojin Xing* and Lei Pan*
*School of Software, Shanghai Jiaotong University, P.R China
†Med-X Research Institute, Shanghai Jiaotong University, P.R Chinaa
Email: gulixu@sjtu.edu.cn

***Abstract*—Reconstructing meshes from 3D medical images is an important but complicated task, which provides a fundamental data structure for Bio-medical applications. In this paper, a novel and easily implemental solution is proposed. Instead of following the popular methods, we sample on the medical images with centroidal Voronoi tessellations, and then build the Delaunay mesh from the sampled points directly. By applying the algorithm on the CT images and evaluating the mesh quality metrics, the solution presents to be good at reconstructing complicated objects and generates optimized meshes, which shows that it is suitable for anatomical structure modeling.**

I. Introduction

Reconstructing tetrahedral meshes from 3D medical images is a non-trivial task in Bio-medical engineering. Many applications, such as finite element analysis on the soft-tissues[1], work on the reconstructed tetrahedral meshes.

The most common solutions in generating tetrahedral mesh are two-phase solutions, which firstly extract the surface using a surface reconstruction algorithm([2], [3]), and then fill the object through a tetrahedral mesh generation algorithm([4]). The key problem is that the human organs are usually so complicated that the surface reconstruction algorithm cannot make a guarantee on building valid surface meshes for the next phase. For example, in [5], the authors have to do lots of complicated work on repairing the extracted surface mesh, which let the solution hard to be implemented in practics.

Another common solutions based on adaptive space-trees. These solutions firstly subdivide the object space into regular geometric structures, such as uniform grids in [6] or Delaunay tetrahedralization in [7], then refine them into tetrahedra and consider the boundary situation to make sure the output mesh respects to the original shape. This kind of solutions is usually quite easy to be understood and implemented, but they also produces poor quality elements on the boundary.

As we known, most of the exist medical applications choose the above two kinds of solutions. But they are not the only ways to generate the tetrahedral meshes from medicial images. We try to solve this problem in another way, which firstly samples all the vertices and then connects them together to generate the last result. There are three reasons make this kind of solutions attractive:

1) The geometric information, which is critical for the representation of the objects, can be computed from 3D medical images through image analysis methods.
2) The quality of the mesh can be improved by optimizing the distributation of the sampling points. It's much more easier to fix a problem, such as holes on the surface, through get a better sampling set than employing complicated mesh repair algorithms.
3) According to [8], a theory guarantee may be existed on generating manifold meshes respected to the original shapes, which is quite critical for medical applications.

For above reasons, the proposed solution firstly samples on the solid object with a global optimized sampling data structure named centroidal Voronoi tessellations, and then reconstructs the tetrahedral mesh from the samplings. The sample points, especially on or near the surface, are took special consideration, so that the properties described in [2] will not be broken, which makes sure the output meshes be in right shapes.

Section II gives the definitions of important concepts. Section III describes the entire solution and Section IV shows the experimental results. Section V concludes this paper and discusses the possible improvements in the future.

II. Definitions

A. Voronoi Tessellations and Delaunay Tetrahedral Mesh

3D *Voronoi tessellations* and the dual *Delaunay tetrahedral mesh* are both important structures in geometric modeling[9]. Let S denote the set of points which called *site*s, the *Voronoi region* corresponding to a site $z_i \in S$ is

$$V_i = \{x \in \mathcal{R}^3 | d(x, z_i) < d(x, z_j), z_j \in S - \{z_i\}\} \quad (1)$$

where $d(x, y)$ is the Euclidean distance between two points. The union of all the Voronoi regions ($\{V_i\}_{i=0}^{k}$) is *Voronoi tessellations*.

Delaunay tetrahedral mesh is the dual of the 3D *Voronoi tessellations*. It can be computed from the *Voronoi tessellations* of S by connecting the sites whose corresponding Voronoi regions are adjacent to each other. The tetrahedra in the Delaunay mesh have 'empty' circumspheres, where the 'empty' means there is no other site of S inside the sphere.

B. Centroidal Voronoi Tessellations

The *centroidal Voronoi tessellations* (*CVT*) is special Voronoi tessellations whose sites are the mass centroids of their corresponding Voronoi regions[10]. Formally to say, give a density function ρ, and a set of sites S, we can get their associated Voronoi regions $\{V_i\}$ and the mass centroids $\{z_i^*\}$

$$z_i^* = \frac{\int_{V_i} y\rho(y)\mathrm{d}y}{\int_{V_i} \rho(y)\mathrm{d}y} \quad (y \in V_i) \tag{2}$$

The *centroidal Voronoi tessellations* is the *Voronoi tessellations* of S when $z_i = z_i^*$.

Centroidal Voronoi tessellations have been employed in many applications[11], include data compression, image processing, and mesh optimization[10]. It has been proved to be a good global optimized sampling method.

C. Local Feature Size and r-Sample

The *local feature size*($lfs(p)$) of a point p is the Euclidean distance from p to medial axis. A surface sampling set S is said to be r-sampled of surface F if for every $p \in F$, there is a point $q \in S$ with $d(p,q) < r \cdot lfs(p)$. *Local feature size* is an important concept for surface reconstruction. In [2], the authors give the CRUST algorithm, which reconstructs the surface from a r-sampled surface point sets, with a theory guarantee on that the generated surface is homeomorphic to the original shape if r is sufficiently small. A groups of algorithms([3]) are derived from their work.

III. CVT-based Meshing Algorithm

A. Overview

The proposed solution includes following phases:

1) Image pre-processing: gets a smoothed binary volume and computes the lfs for each voxel.
2) Surface sampling: samples on the surface to make sure the generated mesh respects to the original objects.
3) CVT-based sampling: gets optimized interior sampling points as the vertices of the tetrahedra.
4) Shallow points elimination: removes the interior sampling points which are too closed to the surface.
5) Mesh generation: generates the tetrahedral mesh from the sampling points described above.

B. Image Pre-processing

The first step is extracting the organs from the 3D medical images. It's another big research topic so we will not discuss it here. In our experiments, we use ITK-SNAP([12]) to get the segmented binary images. A voxel is labeled as 1 if it is inside, otherwise 0.

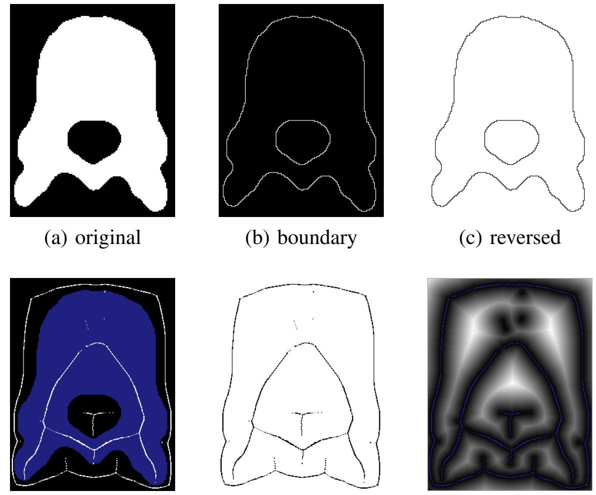
Then the *local feature size* is computed. The general idea is computing the skeleton of the reversed boundary images, and then applying Euclidean Distance Transform(Fig.1). For performance reason, we implement a GPU version of Euclidean Distance Transform based on the work of [13].

C. Surface Sampling

The vertices of the mesh surface(B) is sampled from the boundary voxels F according to the energy function

$$energy(p) = e^{-\frac{d^2(p,q) - r^2 lfs^2(p)}{2\sigma^2}} \tag{3}$$

where $p \in F$ and q is the nearest sampled voxel in B.



(a) original (b) boundary (c) reversed (d) medial axis (e) reversed (f) $lfs(x)$

Fig. 1. Compute lfs. (a) to (d) compute medial axis and (f) gets lfs through applying EDT on (e).

Algorithm 1 keeps the current energy of all the boundary voxels and samples the voxel p with the maximum energy, and updates the energy of the near voxels. The program terminated when B is r-sampled.

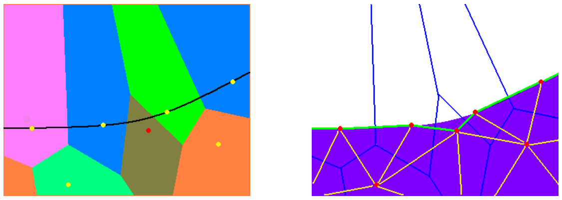
Algorithm 1 Surface Sampling

procedure SurfSample(B, F, r)
  $B \leftarrow \emptyset$
  for all $p \in F$ **do**
    $h(p) \leftarrow lfs(p)$
    $energy(p) \leftarrow e^{-\frac{(1.0 - r^2)\cdot lfs^2(p)}{2\sigma^2}}$
  end for
  while $F \neq \emptyset$ **do**
    $p = p_i \in F$ with max $energy(p_i)$.
    $F = F - \{p\}$
    $B = B + \{p\}$ ▷ p is sampled
    for all $q \in F$ with $d(p,q) < lfs(p)$ **do**
      $h(q) \leftarrow min\{d(p,q), h(q)\}$
      $energy(q) \leftarrow e^{-\frac{h^2(q) - r^2 lfs^2(p)}{2\sigma^2}}$
      if $d(p,q) < r \cdot lfs(q)$ **then**
        $F = F - \{q\}$
      end if
    end for
  end while
end procedure

D. CVT-based Sampling

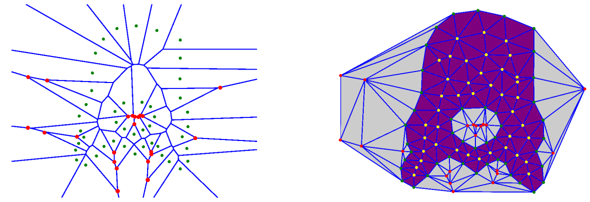
The interior points(I) are sampled according to the density function

$$\rho(p) = \begin{cases} 1.0 - e^{-\frac{lfs^2(p)}{2\sigma^2}} & \text{if inside} \\ 0.0 & \text{if outside} \end{cases} \tag{4}$$



(a) (b)

Fig. 2. (a)Distinguish the shallow points(red) from Voronoi tessellations. (b)The Voronoi facets are not nearly perpendicular to the surface because of the shallow point.



(a) Build Voronoi tessellations from B and get exterior poles EP(red). (b) Build Delaunay mesh from $B \cup EP \cup I$, and remove the elements incident to the poles(gray).

Fig. 3. Mesh generation

based on centroidal Voronoi tessellations. Algorithm 2 is derived from Lloyd's algorithm[11]. In each iteration, it computes the Voronoi tessellations from I, gets the mass centroids(I^*) of the Voronoi regions, and then takes I^* as the new sites for next iteration. This process continuous until the sum of displacement($delta$) is sufficient small.

Algorithm 2 CVT-based Sampling

procedure CVTSAMPLE(I, ρ, ϵ)
  $I \leftarrow$ Random Sampled Points
  $delta \leftarrow \infty$
  while $delta > \epsilon$ **do**
    $V \leftarrow BuildVoronoi(I)$
    $I^* \leftarrow \emptyset$
    $delta \leftarrow 0$
    for all $V_i \in V$ **do**
      $z_i^* \leftarrow \frac{\sum_{p \in V_i} p \cdot \rho(p)}{\sum_{p \in V_i} \rho(p)}$ ▷ mass-centroid z_i^*
      $I' \leftarrow I' + \{z_i^*\}$
      $delta \leftarrow delta + d(z_i^*, z_i)$
    end for
    $I \leftarrow I'$
  end while
end procedure

The Voronoi tessellations are computated on image space based on the vector Euclidean Distance Transform, which labels each voxel to the coordinates of the nearest 0-voxel. In our algorithm, the sampled sites(I) are record as 1-voxels in a volume image. The *BuildVoronoi()* reverses the volume and then applies vector Euclidean Distance Transform on it. Note that the generation of discrete Voronoi tessellations in image space is much faster than the common version in Euclidean space, since the vector EDT can be computed in parallel[13].

E. Shallow Points Elimination

The interior points which are too close to surface will cause trouble when generating the mesh(Section III-F explains why). The shallow points are distinguished from discrete Voronoi tessellations. Travels on all the boundary points $x \in F$, if x belongs to the Voronoi region $V(p), p \in I$, then removes p from I. In Fig.2a, the Voronoi region of the red site includes boundary voxels, so it needs to be removed.

F. Mesh Generation

Now we get the surface and interior sampling points, and need to generate the last result from them. Our mesh generation algorithm is derived from the two-phase surface reconstruction algorithm CRUST[2]. In first phase, the CRUST builds Voronoi tessellations from B, and chooses special Voronoi vertices named poles for each $s \in B$: if s lies on the convex hull, then the pole is the farthest Voronoi vertex of $V(s)$ from s, otherwise the poles will be the furthest Voronoi vertices pair of $V(s)$. The poles can be classified as interior poles (IP) which inside the object, and exterior poles (EP) outside($P = IP \cup EP$). In the second phase, CRUST generates the Delaunay mesh from $B \cup P$, keeps the tetrahedra whose all vertices are in B, and then extracts the boundary triangles as the result surface mesh. The authors proved that if the surface samplings is r-sampled with a sufficiently small r, the generated mesh is in the same topology with the original shape.

The proposed solution is similar to CRUST. It firstly generates the Voronoi tessellations from B, and computes the exterior poles(EP) from the tessellations. And then, it builds the Delaunay mesh from ($B \cup EP \cup I$) and deletes the tetrahedra incident to the poles in EP(Fig.3).

Comparing to CRUST, the proposed solution builds the mesh from $B \cup EP \cup I$, instead of $B \cup P$. We can group I into IP' and $I'(I = IP' \cup I')$. The IP' is the set of nearest points to the IP in CURST. So we get

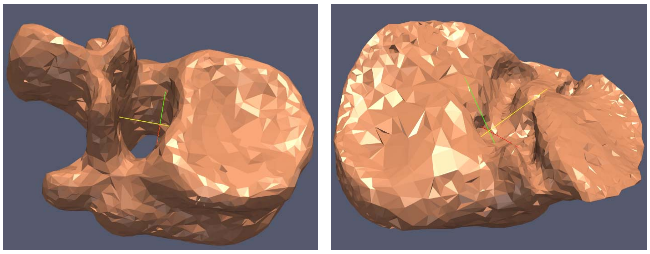
$$B \cup EP \cup I = B \cup (EP \cup IP') \cup I' \tag{5}$$
$$= (B \cup P') \cup I' \tag{6}$$

The above equation can be roughly considered as building the surface from $B \cup P'$, and then inserting the other interior points I'.

[2] proved that the the intersected planes in Voronoi tessellations are nearly perpendicular to the original surface F if B is r-sampled. But in our context, the interior points sampled too closed to the surface, will break this property and make holes on the surface(Fig.2b). So they are removed in III-E.

IV. EXPERIMENTS

We evaluate the quality of the generated mesh through the metrics chosen from Shewchuk[14]. The larger of the value,



(a) bone (b) liver

Fig. 4. Generated meshes.

the better of the element.

a $\frac{V}{r_{mc}^3}$, the metrics of the error bounds between the interpolated function f and the true function $g(\|f-g\|_\infty)$.

b $V(\frac{\sum_{m=1}^{4} A_m}{\sum_{1\le i<j\le 4} A_i A_j l_{ij}^2})^{\frac{3}{4}}$, the metrics of error bounds between the gradient of the interpolated function ∇f and gradient of the true function $\nabla g(\|\nabla f-\nabla g\|_\infty)$.

c $\frac{V}{(\frac{1}{4}\sum_{i=1}^{4} A_i^2)^{\frac{3}{4}}}$, the metrics of the condition number of the stiffness matrix in finite element analysis.

We compare our algorithm to another popular mature solution, which first builds the triangle mesh using marching cubes, and then gets the tetrahedral mesh through the NETGEN[15], with or without the last mesh optimization stage. Table I shows the comparision results of Fig.4a.

TABLE I
COMPARISION RESULT

Metrics		CVT	Not Opt.	Opt.
$\|f-g\|_\infty$	$Q < 0.01(\%)$	2.59	4.21	1.79
	$0.01 \le Q \le 0.4(\%)$	87.02	92.26	66.97
	$Q > 0.4(\%)$	10.39	3.53	31.23
$\|\nabla f-\nabla g\|_\infty$	$Q < 0.05(\%)$	7.66	16.58	0.0
	$0.05 \le Q \le 0.15(\%)$	81.22	70.89	82.84
	$Q > 0.15(\%)$	11.12	0.59	29.11
Cond. Num	$Q < 0.1(\%)$	6.83	7.29	0.0
	$0.1 \le Q \le 0.4(\%)$	91.55	92.67	95.79
	$Q > 0.4(\%)$	1.62	0.04	4.21

The above table shows that the mesh quality generated by the proposed solution is between the other two. The CVT-based sampling algorithm is a global optimized sample algorithm, not a local one. So our solution loses if the compared solution applies a local optimization stage and wins if it doesn't. The mesh optimization is usually considered as an independent post-processing stage of the initial reconstruction. If we include it, the mesh quality will depends on the post-processing, not on the reconstruction itself. The best practics is using our solution to get a good initial reconstruction from medical images and then, if needed, apply local optimization for the worst case guarantee. Note that our generated meshes is Delaunay. It's not difficult to choose a Delaunay-based mesh optimization algorithm.

V. Conclusion And Future Work

We have presented a noval mesh generation solution designed for 3D medical images. This solution computes the *local feature size*, gets the sampling points based on centroidal Voronoi tessellations, and then builds Delaunay mesh from the sampling ponits. The experimental results show that the proposed solution is competent to generate the meshes from the medical images, and global optimized.

There are still more works. One is dealing with the sharp regions which are just in several voxels thickness. In these regions, the medial axises and lfs could not be correctly computed. And we think that a theory guarantee maybe exists since the CRUST gives.

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